Pupin's Theoretical and Experimental Work on Loaded Telephone Lines Accompanied by Modern Full Wave Matrix Approach

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***Abstract*—Pupin's theoretical and experimental work on telephone lines loaded with equally distributed inductance coils of toroidal form is presented in this paper. His technique, known as Pupinization, allowed to significant reduction of attenuation within the voice band making long-distance signal transmission possible in the early days of the telephone. Effects of Pupin's coils inclusion have been enhanced in the paper by introducing so-called Heaviside criterion fulfilment factor and presenting Pupinized line transmission characteristics and bandwidth obtained by using modern full wave matrix approach.**

***Index Terms*—telephone long-distance communication, Pupinization, Heaviside criterion fulfilment factor.**

I. Introduction

Attenuation and dispersion of a modulated signal propagating along a transmission line are caused by the presence of ohmic losses of the line conductors and dielectric between conductors. With dispersion, the propagation velocity depends on frequency which means that various components of the frequency spectrum of the modulated signal propagate with different group velocities. As a result, the time taken for these components to be propagated along a line of given length will not be the same. It would be then impossible to reconstruct the frequency spectrum of a transmitted signal at the end of the line making it completely unrecognizable. This is known as phase distortion and is much more serious than amplitude distortion caused by the frequency dependence of attenuation. Both effects were responsible for impeding the development of telephone voice transmission over long distance in the early days of the telephone.

In 1893 Olive Heaviside developed the electrical circuit theory [1], based on Maxwell's equations, which offers solutions for long distance telephone communications of human voice. Until that time, the transmission line was described by a diffusion equation, which in circuit term, involves a distributed series resistor and parallel capacitor network. Introducing an inductance and taking its proper value into account, Heaviside noticed that the effects of the amplitude and phase distortion both decreased with inductance increase. He derived the so-called *Heaviside criterion*, as a relation between the primary per-unit length parameters of transmission line, $R'C' = L'G'$, for which condition of minimum losses is achieved. In addition, both attenuation constant and velocity of propagation are independent of frequency. Transmission line with such primary parameters is distortionless and it has an infinitive bandwidth assuming that only TEM wave propagates.

From the theoretical work of Heaviside it was known that ordinary telephone lines could not achieve the Heaviside criterion for the ideal signal transmission because they have a small per-unit length inductance ($L'G' << R'C'$). Thus, they are not obviously suitable for the baseband telephone signal transmission. Attempts of many investigators to load such lines with inductance coils connected in series were unsuccessful. It was not before Serbian and American scientist Mihajlo Pupin derived new theory of periodically loaded lines [2-5] and experimentally verified it, first in his laboratory at Columbia University in USA, and then in practice, that this important problem in voice signal transmission was solved. In this connection, Pupin developed the toroidal form of inductance coil without which the theoretical results would have small practical value, because otherwise coils belonging to different circuits would have mutual inductance, and this would result in cross-talk. A toroidal inductance has no external circuits. The Western Electric Co., in New York, and Siemens-Halske of Berlin, developed this type of telephone cable which revolutionized telephonic transmission. In the development of the inductance coils during at least 25 years, Pupin was consulted by both companies.

Over ninety years ago the American Telephone and Telegraph Company established telephonic communications between Boston-New York-Washington. Over cables of this kind, and at that time this was the longest telephonic cable transmission in the world. It was 500 miles or approximately 800 kilometres long. Today the distance has been indefinitely increased by the interposition of vacuum tube amplifiers. But it is admitted that without the inductance coils introduced according to

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Pupin's theory the vacuum tube amplifiers alone would make transmission over telephone cables impossible. Pupin's method of periodically distributed inductive coils, well-known as Pupin loading, is still used in local and trunk telephone lines as it could significantly reduce the attenuation of signals within the voice band (i.e., at frequencies less than 4 kHz). It can be shown that at frequencies for which wavelength λ is significantly greater than the spacing a of the loading coils ($\lambda >> \pi a$ [3,6]), Pupinized line is behaved as an equivalent line with L increased continuously along line.

This paper represents a research work, started in 2004 when the 150th anniversary of Mihajlo Pupin's birth was celebrated [7,8] and continued after that [9,10], established with the main goal to enlighten the scientific achievements of Mihajlo Pupin. Among them, the developed approach of telephone lines loading in series with discrete coils place at regular intervals along a line to allow for long-distance voice transmission was Pupin's greatest success. Pupin's theory of transmission line with periodical inductance loading and its experimental verification conducted trough artificially constructed line are described in detail in this paper. At the same time, a modern full wave matrix approach for transmission line analysis is presented in order to further illustrate the effects of Pupin's coils inclusion on line transmission characteristics and bandwidth. For that purposes, so-called Heaviside criterion fulfilment factor is used and comparison with unloaded line and equivalent uniform line characteristics is presented.

II. Pupin's Theoretical Approach to Current Propagation Through Telephone Lines

Pupin started from mathematical Lagrange solution for vibrations of heavy flexible inextensible string connected to tuning-fork at one end and fixed at the other end. He developed a new mathematical theory of oscillation travelling through string loaded by weights distributed uniformly along its length. (Fig.1a). He considered the mechanical vibration of such string (Fig.1b) as a perfect analogy to the propagation of electric waves in a long wire conductor loaded with periodically inserted inductive coils (Fig.1c). The existence of this analogy is due to the physical fact that the three reactions which accompany the string vibration: the acceleration reaction, the tensional reaction and the frictional reaction follow the same laws as the three reactions which accompany the flow of the variable current in a long wire conductor: the ohmic reactance reaction, inductance reaction and the capacity reaction. Lagrange solution suggested that inductive coils should be inserted at such intervals along the telephone line that for all components of alternating currents (AC) there are a few coils per wavelength. For currents at these wavelengths, this loaded telephone line behaved as an equivalent line with uniformly distributed inductance.

A. The Equations of Propagation through Uniform Line

In [2] Pupin presented his theory of electrical oscillations propagation in linear conductor of length l with uniformly distributed inductance, resistance and capacitance (Fig.2).

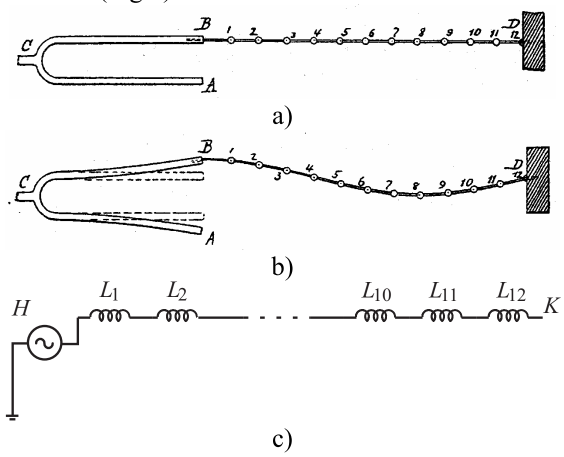


Fig.1. a) String loaded by weights distributed uniformly along its length, b) waves set up in the string when applying forced vibrations under the action of the tuning-fork in a medium which offers appreciable resistance to the string vibration, c) slow-speed conductor (so-called reactance conductor).

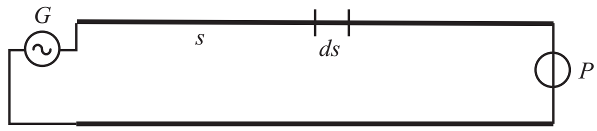


Fig.2. Uniform line supplied at one end by generator G and terminated at the other end by load P (total circuit length is $2l$).

In his definition of uniform line, Pupin started from per-unit-length parameters: L', R' and C'. A line element of length ds with distance s from generator has current i and potential U. According to law of equality of opposing forces (forces of action and reaction), sum of impacts on element ds is zero which Pupin expressed through two differential equations:

$$\left(L'\frac{di}{dt}+R'i+\frac{\delta V}{\delta s}\right)ds=0 \tag{1}$$

$$C'\frac{dV}{dt}=-\frac{\delta i}{\delta s} \tag{2}$$

Differentiating (1) with respect to t and (2) with respect to s, the following differential equation for current can be obtained:

$$L'\frac{d^2 i}{dt^2}+R'\frac{di}{dt}-\frac{1}{C'}\frac{\delta^2 i}{\delta s^2}=0 \tag{3}$$

This equation represents the equality of action and reaction at every point on the line when uniformity is preserved. Insertion of generator and load causes discontinuities on the line for which new equations, describing boundary conditions on both line ends should be written. Pupin gave a general solution for current on such terminated line as:

$$i=\left[K_1\cos(m\xi)+K_2\sin(m\xi)\right]e^{kt} \tag{4}$$

where $\xi=l-s$ and co-ordinate origin starts at the load. Equation (4) satisfies (1) if:

$$-m^2=kC'(kL'+R') \tag{5}$$

In order to solve (4), constants K_1, K_2, and k have to be determined to satisfy boundary conditions at the generator and the load. As a general solution, Pupin considered one that allows arbitrary choice of generator and load. To determine these constants Pupin introduced (6) as:

$$\begin{aligned}
&K_1 = \frac{2mkC'E}{F}, \quad K_2 = \frac{kC'Eh_1}{F} \\
&F = \left(h_0 h_1 - 4m^2\right)\sin(ml) + 2m(h_0 + h_1)\cos(ml) \\
&h_0 = kC'\left(kL_0 + R_0 + \frac{1}{kC_0}\right) \\
&h_1 = kC'\left(kL_1 + R_1 + \frac{1}{kC_1}\right) \\
&D_0 = kC'E
\end{aligned} \tag{6}$$

where: R_0, L_0, C_0, and R_1, L_1, C_1, are impedance parameters of generator and load, respectively, whereas E is a real part of excitation voltage $Ee^{j\omega t}$ applied at the start of the line. Then, (4) transforms to:

$$i = \left[2m\cos(m\xi) + h_1 \sin(m\xi)\right]\frac{D_0 e^{kt}}{F} \tag{7}$$

This general solution for current along the line Pupin then applied to calculate free and forces vibrations on the transmission line terminated at its ends with different impedance parameters of generator and load.

In a modern approach, equations of propagation through uniform line in steady-state condition for sinusoidal excitation can be expressed as:

$$\begin{bmatrix} U_{in} \\ I_{in} \end{bmatrix} = \begin{bmatrix} \cosh(\gamma\xi) & Z_c \sinh(\gamma\xi) \\ \dfrac{\sinh(\gamma\xi)}{Z_c} & \cosh(\gamma\xi) \end{bmatrix} \begin{bmatrix} U_{out} \\ I_{out} \end{bmatrix} \tag{8}$$

whereas so-called secondary parameters of the line, γ (propagation constant) and Z_c (characteristic impedance) are related to primary per-unit length line parameters:

$$\gamma(\omega) = \alpha(\omega) + j\beta(\omega) = \sqrt{(R' + j\omega L')(G' + j\omega C')} \tag{9}$$

$$Z_c = \left|Z_c(\omega)e^{j\varphi_c(\omega)}\right| = \sqrt{(R' + j\omega L')/(G' + j\omega C')} \tag{10}$$

In (8) subscript *in* is related to voltage and current at the start of the line and subscript *out* is related to voltage and current at the end of the line. Distance ξ is measured from the end of the line. It can be noticed that Pupin did not define the characteristic impedance in his approach and his solution had time-space dependence, also in steady-state condition for sinusoidal excitation.

For the special case, when impedances of generator and load are negligible, Pupin found a solution for current in the form

$$\begin{aligned}
&i = -\frac{j\omega C' E\cos(m\xi)e^{j\omega t}}{2m\sin(ml)} \\
&m^2 = \omega^2 L'C' - j\omega R'C'
\end{aligned} \tag{11}$$

where notation of some quantities are changed according to the modern notation for the purpose of comparison. From (8) current at distance ξ from the end of the line can be obtained from the condition U_{out}=0:

$$I_\xi = I_{out}\cosh(\gamma\xi) \tag{12}$$

or, as voltage at the start of the line can be expressed as $U_l = Z_c I_{out}\sinh(\gamma l)$:

$$I_\xi = \frac{U_l \cosh(\gamma\xi)}{Z_c \sinh(\gamma l)} \tag{13}$$

Having in mind that Pupin's factor m is $m=j\gamma$; the following equivalences can be established: $\cos(m\xi) = \cosh(\gamma\xi)$, $\sin(m\xi) = -\sinh(\gamma\xi)$ and $\frac{-j\omega C'}{m} = \frac{1}{Z_c}$. Therefore, (11) and (13) are identical for the condition: $U_l = E/2$. It is interesting that Pupin calculated a mean value of the square of the current at some point on the line as:

$$M\left(\eta^2\right) = \frac{2}{T}\int_0^{T/2} \eta^2 dt \tag{14}$$

probably, because he measured the effective value of current and he was able to solve (14) in closed form substituting a real part of current from (11) in (14).

One of the important conclusions Pupin derived from examples he analyzed was that every current wave can be decomposed into two components: amortized current wave travelling toward the generator and wave travelling toward the load. If attenuation on the line is small, Pupin concluded that a complete standing wave would exist on the line. As mentioned before, Pupin did not define characteristic impedance in his approach and he considered current behaviour under conditions of great reflection from the load as the most important. In addition he introduced term *propagation efficiency*. Pupin stated that it is important not to establish just a small per-unit-length resistance but also to make quantity $\omega L'/R'$ greater. Equivalent to the analysis of circuit with simple AC current, he defined power factor as $\cos\theta$ where $\tan\theta = \omega L'/R'$. In the case of simple circuits with losses one should try to provide that reactive power is significantly higher from active power because, as Pupin said, *this accumulated energy is returned to the generator in the case of simple circuit and propagates in the case of long wires*. In addition, Pupin emphasized the difference between simple circuit analysis and current propagation along the transmission line because in the latter case maximum of delivered power to the load is required.

B. Pupinized Line

In [2,3] Pupin also presented two types of so-called Pupinized line. Pupinized line of the first type consists of coils connected in series and condensers connecting opposite points of the line (Fig.3a) or of similarly connected coils while the condensers joint the points between the consecutive inductances to the ground (Fig.3b). Such constructed line was described by Pupin as *a slow-speed conductor* because the velocity of propagation along such a conductor is smaller than ordinary lines. He then derived the appropriate equations of propagation for this type of Pupinized line.

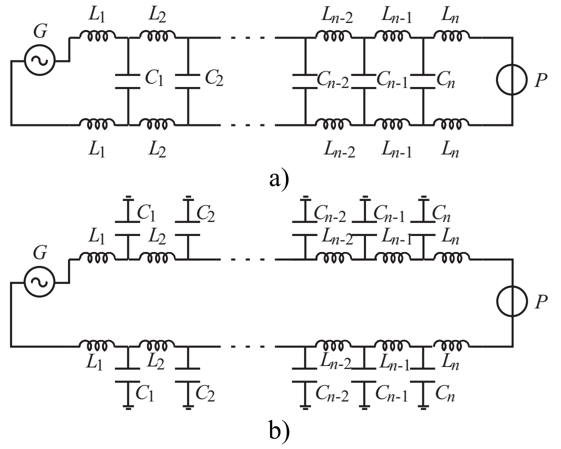


Fig.3. a) Slow-speed conductor of the first type, b) modified slow-speed conductor of the first type.

Circuit shown in Fig.3a is divided into sections. All sections, except the first and last section due to presence of generator and load, are identical in the sense that they have an equal resistance, capacitance and inductance. In order to find a solution, Pupin assumed a sinusoidal excitation and applying the law of equality of action and reaction he formed the following system of differential equations for each section including generator and load circuits:

$$
\begin{aligned}
&(L_0+2L)\frac{di_1}{dt}+(R_0+2R)i_1+P_1+P_0=Ee^{j\omega t}\\
&2L\frac{di_2}{dt}+2Ri_2+P_2-P_1=0\\
&-\quad-\quad-\\
&2L\frac{di_{n-1}}{dt}+2Ri_{n-1}+P_{n-1}-P_{n-2}=0\\
&(L_1+2L)\frac{di_n}{dt}+(R_1+2R)i_n-P_{n-1}+P'=0
\end{aligned}
\tag{15}
$$

where: L, R and C are appropriate values of coils and condensers in each section. First and last equations are related to generator and load circuit, respectively, whereas other equations are related to the typical section (with appropriate indices changes). Parameter P_i represents potential on i-th condenser ends whereas parameters P_0 and P' represent potential difference between generator and load ends, respectively. For sinusoidal steady-state response, using (16), Pupin reduced the system of differential equations (15) to system of linear equations with complex coefficients and n unknown currents.

$$
i_1-i_2=C\frac{dP_1}{dt},\quad i_2-i_3=C\frac{dP_2}{dt},\ldots \tag{16}
$$

Using two approaches: direct method and indirect method of successive elimination, he then managed to find solution for all currents concluding that he found a complete problem solution. He used a similar approach for the circuit in Fig.3b.

Pupin turned his attention to the question under which condition (except $n\to$oo) such Pupinized line behaved as an equivalent line. On the particular example of Pupinized line with coils and condensers values: L=0.0125 H, C=0.025 μF and R=2.5 Ω and at ω=3000 rad/s, he showed that the expression for current in each section in Fig.3a can be reduced to (7) if a number of section is n=200 (400 coils). Pupin concluded that up to a frequency of 1 kHz, such a line, shown in Fig.3, represents very nearly an ordinary telephone line with length of 1000 miles having per mile an inductance of 0.005 H, a resistance of 1 Ω and a capacitance of 0.01 μF; but even for a frequency of 3.5 kHz a slow-speed conductor represents, if not very nearly, then quite approximately with an error of 1-2 %, an ordinary line with uniformly distributed inductance, resistance and capacitance. Such high-inductance line has not only the advantage of small attenuation but also of very small distortion as all frequencies in human voice are attenuated to similar level. He then evolved the following general rule: *If n is the number of coils per wavelength, then for that wavelength the slow-speed conductor will represent an ordinary telephone line with the accuracy of the equation*:

$$
\sin\frac{\pi}{n}\approx\frac{\pi}{n} \tag{17}
$$

Much convenient expression Pupin derived from an introduction of what he called *angular distance* between two points on a line conductor. Two points at a linear distance of the wavelength have an angular distance of 2π. With this understanding it follows that these points at a linear distance of λ/n will have an angular distance of $2\pi/n$. Then the rule given above can now be stated as: *A slow-speed conductor resemble an ordinary line conductor with a degree of approximation measured by the ration of the sine of half the angular distance covered by a coil to half the angular distance itself.* This rule is the foundation upon which this Pupin's invention was based on. He confirmed such theoretical conclusions by experiment described in the following section.

After that, it was an easy matter for Pupin to pass on to a Pupinized line of the second type, better adapted to commercial use for the purpose of diminishing the attenuation of electrical waves. The slow-speed conductor of the second type, that Pupin called *a reactance conductor*, (Fig.1c, H is the transmitting and K the receiving end of a long electrical conductor) differs from those in Fig.3 as it has a distributed capacitance only in place of the lumped capacitance and, in addition to the lumped inductance and resistance, it has evenly distributed inductance and resistance. Therefore, the *reactance conductor* is a long electrical wire conductor having preferably equal reactance (inductance) sources in series at preferably equal intervals. It is reasonable to expect that reactive slow-speed conductor of the second type will operate like an ordinary uniform line of the same inductance, capacitance and resistance per unit length under the same condition under which the slow-speed conductor of the first type so operates.

For telephony, the angular distance between any two successive coils should sufficiently satisfy the previously given rule for the highest frequency, which is of importance in the telephonic transmission of speech. Pupin suggested that in order to transmit speech telephonically over a wire stretched upon poles, a sufficiently-high degree of approximation to a uniform telephone line would be obtained if fifteen coils per wavelength of the highest frequency was used This mean one coil per mile. In the case of telephone transmission over a submarine cable, Pupin thought that it would be necessary to introduce sixteen coils per wavelength or eight coils per mile. In addition, when interpolated reactance sources consist of simple coils, Pupin suggested that they should be made, preferably, without iron cores, so as to avoid as mush as possible hysteresis and Foucault-current losses and current distortion. If for any special reasons coils of small dimensions per unit inductance are required, then iron or preferably the finest quality of steel should be employed and the magnetization kept down as much as possible [3].

III. PUPIN'S EXPERIMENTAL WORK

In order to verify his theory, Pupin constructed an artificial line on which he investigated, in laboratory conditions, current propagation along telephone lines. Artificial slow-speed line consisted of 24 coils connected in series. Construction of used coil without iron cores is described in Fig.4 [2].

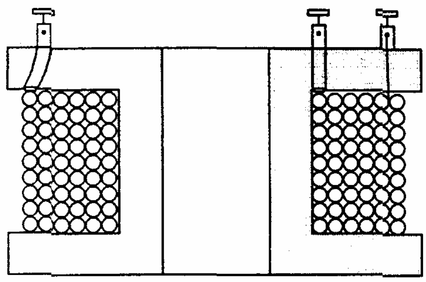
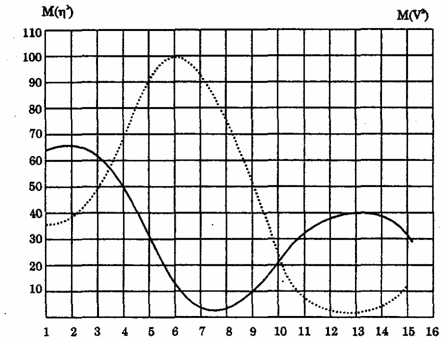


Fig.4. Construction of coil used for artificially built line.

Certain number of layers of wire was wound on wooden spools. The height of layer of wire was 3 inches, diameter of inside layer was 7 inches and the number of layers of wire was 8. Each layer before winding was covered by paraffin paper. Silver foil (staniol metal) was used as a wrapping material around layer. Such foil was then covered with the paraffin paper to allow the next wire layer winding. All silver foils were connected in series. Each coil had the following parameters: L=0.05 H, C=0.1 μF and R=10 Ω which was equivalent to the telephone line of first class (used between New York and Chicago) with length of 10 miles. Losses in such coil were higher than on ordinary telephone lines but for generator under 300 V they did not have significant influence of line behaviour.

On such experimental line with the total length of 240 miles Pupin verified his theory presented in section II. He measured the effective values of voltage and current at some characteristic points on the line excited with source of different voltage and frequency (V_{gen}=60÷234 V, f=25÷750 Hz). As a result, he obtained curves, shown in Fig.5 [2], confirming existence of standing waves on the line. He used the distance between maximum and minimum of the current to experimentally determine the wavelength. From the relation $v=\lambda/f$, he calculated the velocity of wave propagation on the line and compared that value with theoretically calculated velocity from relation $v=\omega/\beta$. Excellent agreement between these two calculated velocities was achieved. It should be pointed out that Pupin determined a phase coefficient β from primary line parameters using the following equation:

$$\beta = \sqrt{\frac{\omega C'}{2}\left[\sqrt{R'^2+\omega^2 L'^2}+\omega L'\right]} \qquad (18)$$



	M(η²)	M(V²)
1	68	173
2	67	190
3	61	248
4	51	350
5	37	450
6	13	490
7	4	470
8	3.5	375
9	5.5	252
10	20	110
11	32	40
12	38	16
13	41	14
14	40	70
15	35	
16		

Fig.5. Curves of average values of square of the current (solid line) and square of the voltage (dotted line) on the line without load (V_{gen}= 234 V, f=610 Hz).

Besides that artificial line, Pupin constructed the other line according to the Fig.3a [2]. Such line consisted of 400 coils connected in series. One part of two rows of coils is presented in Fig.6 (coil 1,2,3 …). Coils of each row were connected to the same metal pipe. Pipes are marked as A and B. For each wire connecting two successive coils, an additional wire is attached to it in order to connect one condenser. Condensers are marked as I, II, III … in Fig.6.

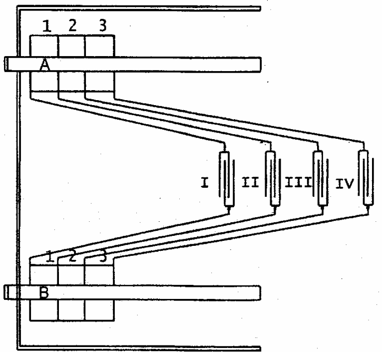


Fig.6. Artificial line constructed according to Fig.3a.

Each coil had the following parameters: L=0.0125 H, R=2.5 Ω. Capacity of each condenser was 0.025 μF. Each pair of coil and condenser corresponded to the telephone line of length 2.5 miles. Two rows of 40 coils each with appropriate number of condensers were placed into a box representing a line with a length of 200 miles. Five such boxes simulated a line with a total length of 1000 miles. At frequencies up of 750 Hz, Pupin obtained curves M(η²) and M(V²) with the same shape as shown in Fig.5 confirming that such Pupinized line was equivalent to uniform telephone line with length of 1000 miles.

Similarly to this artificial line, Pupin made another line having 250 sections [4]. Each section consisted of one paraffin paper A (Fig.7) having on each side metal stripe ab with the following parameters: L=0 H, C=0.074 μF and

R=9 Ω. All sections were connected in series corresponding to the cable with length of 250 miles having per-unit-length capacitance of 0.074 μF/mile and resistance of 9 Ω/mile. Cable sections were grouped into five boxes with a total size of ¼ m^3. At each mile Pupin was able to insert a special inductive coil structure (Fig.8) with total inductance of 0.058 H. Without coils that was a ordinary unloaded line. He compared propagation along such loaded and unloaded line and confirmed experimentally equivalence between non-uniform and appropriate uniform conductor up to 625 Hz.

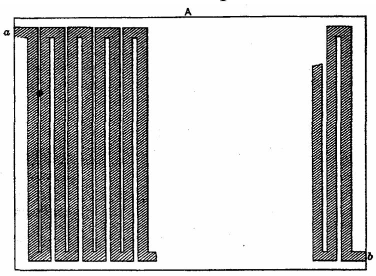


Fig.7. Section layout.

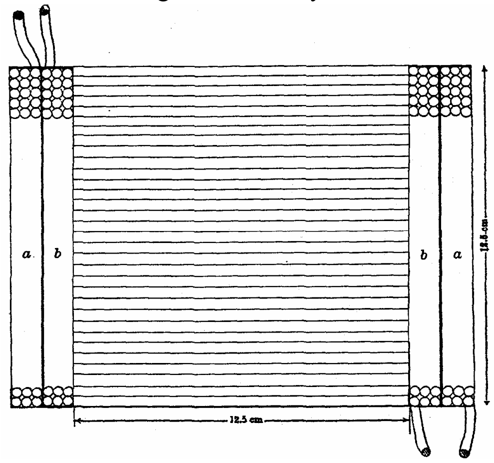


Fig.8. Construction of coil inserted at each mile.

IV. TRANSMISSION LINE PARAMETERS

In telecommunications, a modulated signal is regularly propagated along the line, occupying a considerable frequency band. Two quantities can be used to define its velocity of propagation: the phase velocity, v_p, as a velocity of propagation for the carrier and the group velocity, v_g, as a velocity of propagation for the envelope of modulated signal or velocity with which energy is propagated along the line:

$$v_p(\omega)=\frac{\omega}{\beta(\omega)}, v_g(\omega)=\frac{d\omega}{d\beta(\omega)} \tag{19}$$

In reality, function $\beta(\omega)$ is not a straight line so that v_p and v_g are generally different from each other. Also, they vary with frequency. Since the group velocity represents velocity of propagation of the various components in the modulated signal frequency spectrum, the time taken for the components to be propagated along a line of given length will not be the same. Then, it would be impossible to reconstruct the spectrum of transmitted signal at output which leads to the so-called signal phase distortion. Such an effect, known as dispersion, is a great problem in telecommunications and it can be overcome by inserting expensive variable delay lines in each frequency band. For unit length line, the phase and group delay are related to the phase and group velocity as:

$$\tau_p(\omega)=\frac{1}{v_p(\omega)}=\frac{\beta(\omega)}{\omega}, \tau_g(\omega)=\frac{1}{v_g(\omega)}=\frac{d\beta(\omega)}{d\omega} \tag{20}$$

Of possible interest for an analysis are values of transmission line parameters for $\omega\to 0$ and $\omega\to\infty$. To determine them, we start from the product of phase and group delay:

$$\tau_p(\omega)\tau_g(\omega)=\frac{\beta(\omega)}{\omega}\frac{d\beta(\omega)}{d\omega}=\frac{1}{2\omega}\frac{d\beta(\omega)^2}{d\omega}=\tau_{pg}(\omega) \tag{21}$$

Solving a system of two equations, obtained by equating the real and imaginary part of the left and right side of (9), for unknown $\alpha(\omega)$ and $\beta(\omega)$, the following expression for $\beta(\omega)$ can be obtained:

$$\beta^2(\omega)=\frac{\omega^2L'C'-R'G'}{2}+\sqrt{\left(\frac{\omega^2L'C'+R'G'}{2}\right)^2+\left(\frac{\omega(L'G'-R'C')}{2}\right)^2} \tag{22}$$

Finding the derivation $d\beta(\omega)/d\omega$, product of phase and group delay can be expressed as:

$$\tau_{pg}(\omega)=\frac{1}{4}\left(2L'C'+\frac{2\omega^2L'^2C'^2+L'^2C'^2+R'^2C'^2}{\sqrt{\left(R'^2+\omega^2L'^2\right)\left(G'^2+\omega^2C'^2\right)}}\right) \tag{23}$$

Using L'Hopital's rule and the previous equation, the value of phase delay for $\omega\to 0$ is:

$$\tau_p(\omega)_{|\omega\to 0}=\sqrt{\frac{d\beta^2/d\omega}{d\omega^2/\omega}}_{|\omega\to 0}=\sqrt{\tau_{pg}(\omega)_{|\omega\to 0}} \tag{24}$$

$$\tau_p(\omega)_{\omega\to 0}=\frac{(L'G'+R'C')/2}{\sqrt{R'G'}}=\sqrt{L'C'}\frac{(L'G'+R'C')}{2\sqrt{R'G'L'C'}} \tag{25}$$

Applying $\omega\to 0$ on (21) and using (24) it can be shown that the values of phase delay and group delay are equal for $\omega\to 0$:

$$\tau_g(\omega)_{\omega\to 0}=\frac{(L'G'+R'C')/2}{\sqrt{R'G'}}=\sqrt{L'C'}\frac{(L'G'+R'C')}{2\sqrt{R'G'L'C'}} \tag{26}$$

Similarly, the values of phase and group delay for $\omega\to\infty$, can be found as:

$$\tau_p(\omega)_{|\omega\to\infty}=\tau_g(\omega)_{|\omega\to\infty}=\sqrt{L'C'} \tag{27}$$

The values of phase constant and characteristic impedance for these two end frequencies can be easily obtained as:

$$\beta(\omega)=\begin{cases}0, & \omega\to 0\\ \infty, & \omega\to\infty\end{cases}, Z_c(\omega)=\begin{cases}\sqrt{R'/G'}, & \omega\to 0\\ \sqrt{L'/C'}, & \omega\to\infty\end{cases} \tag{28}$$

To determine the limit values of attenuation, the following relation between attenuation and phase constant is used

$$\alpha(\omega)=\frac{\omega(L'G'+R'C')}{2\beta(\omega)}=\frac{(L'G'+R'C')/2}{\beta(\omega)/\omega} \tag{29}$$

Having in mind that denominator in last equation represents phase delay whose values for two end frequencies are already determined by (25) and (27), the values for attenuation constant for $\omega\to 0$ and $\omega\to\infty$ are

$$\alpha(\omega)=\begin{cases}\sqrt{R'G'}, & \omega\to 0\\ \sqrt{R'G'}\dfrac{(L'G'+R'C')/2}{\sqrt{R'G'L'C'}}, & \omega\to\infty\end{cases} \quad (30)$$

V. Heaviside Criterion Fulfilment Factor

Ratio of per-unit length inductance of the unloaded line *L'* and required Heaviside's per-unit length inductance L_H', obtained from Heaviside criterion ($L_H'=R'C'/G'$), is introduced and named as Heaviside criterion fulfilment factor for unloaded line:

$$l = L'/L_H' \quad (31)$$

The values of characteristic propagation functions for two end frequencies $\omega\to 0$ and $\omega\to\infty$ as a function of factor *l* as well as their values for *l*=1 are given in Table 1.

Table 1: Values of characteristic propagation functions for $\omega\to 0$, $\omega\to\infty$ and Heaviside criterion (*l*=1)

Propagation functions	$\omega\to 0$	$\omega\to\infty$	Heaviside condition (*l*=1)
$\beta(\omega)$	0	∞	$\beta_H=\omega\sqrt{L_H'C'}$
$\alpha(\omega)/\alpha_H$	1	$\dfrac{(l+1)/2}{\sqrt{l}}$	$\alpha_H=\sqrt{R'G'}$
$\lvert Z_c(\omega)\rvert/Z_{cH}$	1	$\sqrt{l}$	$Z_{cH}=\sqrt{R'/G'}$
$\tau_p(\omega)/\tau_{pH}$	$(l+1)/2$	$\sqrt{l}$	$\tau_{pH}=\sqrt{L_H'C'}$
$\tau_g(\omega)/\tau_{gH}$	$(l+1)/2$	$\sqrt{l}$	$\tau_{gH}=\sqrt{L_H'C'}$

In order to graphically illustrate the areas of changes of characteristic propagation functions with frequency, new functions have been introduced and shown in Fig.9:

$$f_a(l)=\frac{l+1}{2},\ f_g(l)=\sqrt{l},\ f_k(l)=\frac{f_a(l)}{f_g(l)}=f_k(1/l) \quad 32)$$

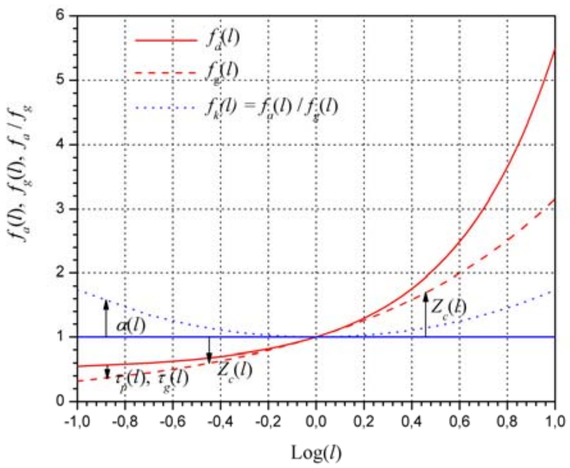


Fig.9. Areas of changes of characteristic propagation functions with frequency versus factor *l*.

The telephone lines have a small per-unit length inductance and as a result, factor *l* is, in reality, always smaller than 1 (even with inserted Pupin coils) but for the purpose of theoretical analysis, the values of *l* bigger than 1 are taken into account as well. From the Fig.9 it can be noticed that the increase of factor *l* narrows the area of possible changes of attenuation and characteristic impedance as well as phase and group delay with frequency. This behaviour is valid until *l* equal to 1 and then there is a reversed proces with the further increase of *l*. In addition, around *l*=1, the changes of these parameters with frequency can be neglected. From the Table 1 and Fig.9 the following relations which identify the areas of changes of attenuation constant, characteristic impedance, phase and group delay with factor *l* are:

$$1\le \alpha(\omega)/\alpha_H \le f_k(l) \quad (33)$$

$$\begin{gathered} f_g(l)\le \lvert Z_c(\omega)\rvert/Z_{cH}\le 1,\ l\le 1\\ 1\le \lvert Z_c(\omega)\rvert/Z_{cH}\le f_g(l),\ l\ge 1\end{gathered} \quad (34)$$

$$f_a(l)\ge \tau_{p(g)}(\omega)/\tau_{p(g)H}\ge f_g(l) \quad (35)$$

Dependance of attenuation constant, characteristic impedance, phase and group delay, normalized with their value at fulfilled Heaviside criterion, for different values of factor *l* are shown in Figs.10-13, respectively. Primary parameters used for calculation are: *R'*=14.2 Ω/mile, *C'*=138 nF/mile and *G'*=24 µS/mile. Inductance required for fulfillment of Heaviside criterion is obtained as $L_H'=R'C'/G'$=0.08165 H/mile. The value of factor *l* is changed by increasing *L'*.

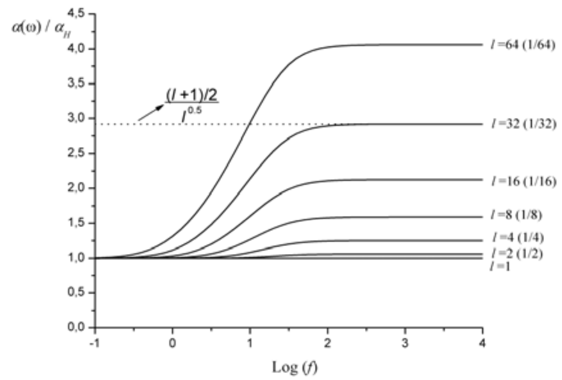


Fig.10. Normalized function $\alpha(\omega)$ for different values of factor *l*.

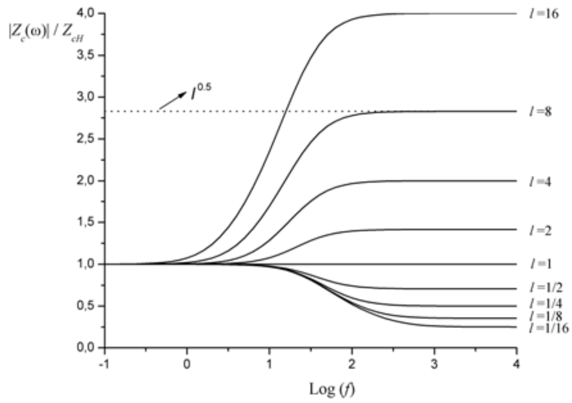


Fig.11. Normalized function $Z_c(\omega)$ for different values of factor *l*.

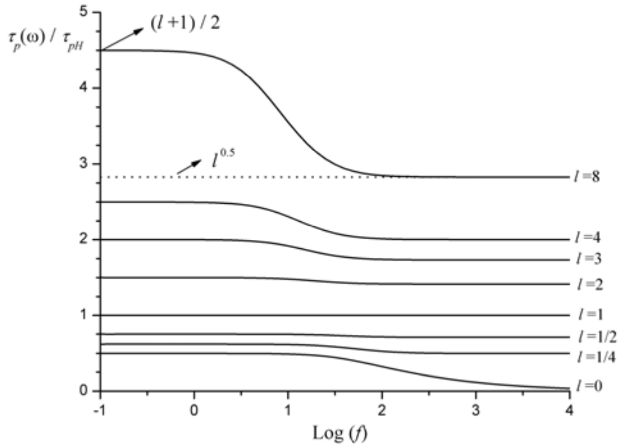


Fig.12. Normalized function $\tau_p(\omega)$ for different values of factor l.

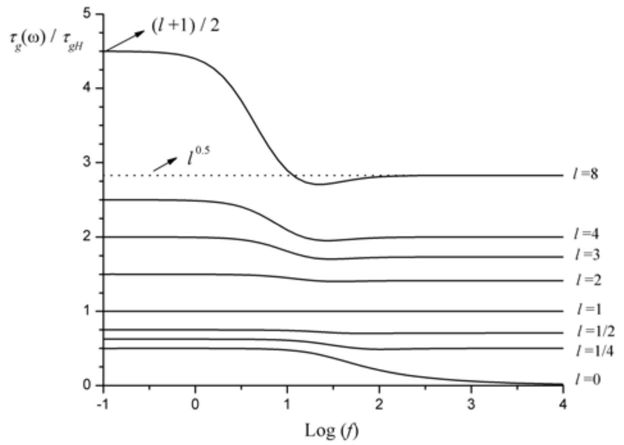


Fig.13. Normalized function $\tau_g(\omega)$ for different values of factor l.

From presented figures it can be estimated for how much per-unit length inductance should be increased (e.g. by inserting Pupin coils) to keep changes of attenuation, phase and group delay in acceptable limits from the signal propagation dispersion point of view.

With an insertion of discrete coils, placed at regular intervals along the line, a total per-unit length inductance of such obtained Pupinized line can be expressed as:

$$L_t' = L' + \frac{L_{pc} n}{\lambda_{\min}} \tag{36}$$

where $n=\lambda_{\min}/a$, represents the number of equally inserted discrete coils per wavelength $\lambda_{\min}$ corresponding to the maximum frequency to be considered for telephone communication. If the highest frequency $f_{\max}$ to be transmitted is specified, the corresponding wavelength is:

$$\lambda_{\min} = \frac{2\pi}{\beta(\omega_{\max})} \tag{37}$$

where β is a phase constant whose value at specified frequency can be easily found from (22). For Pupinized line of total per-unit length inductance L_t', the inductance of discrete coils, required to achieve this increase of per-unit length inductance for different values of n can be found from (36) as:

$$L_{pc} = \lambda_{\min}(L_t' - L')/n = \lambda_{\min} L_H'(l_t - l)/n \tag{38}$$

where Heaviside criterion fulfilment factor of periodically loaded line is defined similarly to the case of unloaded line:

$$l_t = L_t'/L_H' \tag{39}$$

Because of coil losses, the insertion of discrete coils changes per-unit length resistance of the line as well. Thus, the required Heaviside's per-unit length inductance L_H' for Pupinized line can be found as $L_H'=(R'+R_{pc}/a)C'/G'$.

VI. Modern Full Wave Matrix Approach

Periodically loaded transmission line of length d consists of a number of sections as shown in Fig.14.

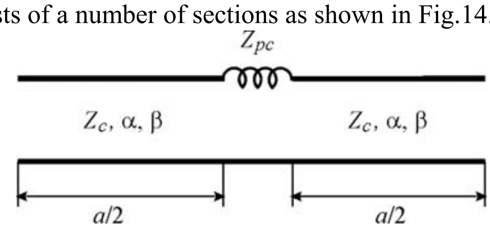


Fig.14. Discrete inductance coil connected in series with two sub-sections of lossy transmission line.

Discrete coil is assumed to be placed between two line sub-sections of length $a/2$. Secondary parameters of each sub-section are: $\gamma=\alpha+j\beta$ and Z_c. Impedance of discrete coil is $Z_{pc}(\omega)=R_{pc}+j\omega L_{pc}$. Such Pupinized line generally can not be considered as an equivalent line with the same primary parameters per unit length and has to be treated on the following way. Wave $[T_a]$ matrix of a periodically loaded line section is the product of three matrices.

$$[T_a] = [T_{a/2}][T_{pc}][T_{a/2}] \tag{40}$$

where

$$[T_{a/2}] = \begin{bmatrix} e^{\gamma(\omega)a/2} & 0 \\ 0 & e^{-\gamma(\omega)a/2} \end{bmatrix} \tag{41}$$

and

$$[T_{pc}] = \frac{1}{2}\begin{bmatrix} 2 + Z_{pc}/Z_c & -Z_{pc}/Z_c \\ Z_{pc}/Z_c & 2 - Z_{pc}/Z_c \end{bmatrix} \tag{42}$$

Full wave $[T]$ matrix of such Pupinized line, represented as a cascade connection of sections shown in Fig.14 can be obtained as:

$$[T] = \prod_{i=1}^{N-1} [T_a]_i \tag{43}$$

where N is the number of sections, $N=d/a$. To take into account the terminating impedances, Z_p, on each side of Pupinized line (Fig.15), additional $[T_{pi}]$ matrices (i=1,2), given by (44) are required.

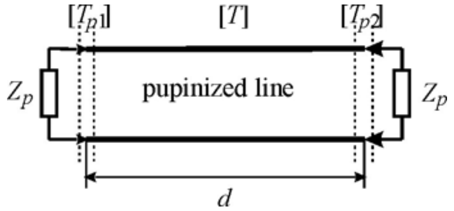


Fig.15. Terminated Pupinized line.

$$[T_{pi}] = \frac{1}{2Z_{c2}} \begin{bmatrix} Z_{c2} + Z_{c1} & Z_{c2} - Z_{c1} \\ Z_{c2} - Z_{c1} & Z_{c2} + Z_{c1} \end{bmatrix} \quad (44)$$

where: $Z_{c1}=Z_p$, $Z_{c2}=Z_c$ for $i=1$ and $Z_{c1}=Z_c$, $Z_{c2}=Z_p$ for $i=2$. Full wave $[T_t]$ matrix of terminated Pupinized line is then:

$$[T_t] = \begin{bmatrix} T_{11t}(\omega) & T_{12t}(\omega) \\ T_{21t}(\omega) & T_{22t}(\omega) \end{bmatrix} = [T_{p1}][T][T_{p2}] \quad (45)$$

In finding frequency dependence of transmission coefficient for the Pupinized line, the following relation has to be used for all frequencies:

$$S_{21}(\omega) = \frac{1}{T_{11t}(\omega)} \quad (46)$$

Transmission coefficient of unloaded transmission line (without discrete coils) can also be obtained from (46) but putting $Z_{pc}=0$ in (42).

It is interesting to compare the transmission and bandwidth of Pupinized line with an equivalent uniform line that has the same primary parameters per unit length: $R'+R_{pc}/a$, $L'+L_{pc}/a$, G' and C'. Following Pupin's rule, as long as the frequencies on the line are not too close to frequencies for which wavelength would be of the order as the spacing a of the loading coils, the Pupinized line can be treated as equivalent line obtained as if inductance was increased continuously along the line [6]. For the equivalent uniform line, the secondary parameters are given as:

$$\gamma_{eq} = \sqrt{[(R'+R_{pc}/a) + j\omega(L'+L_{pc}/a)](G'+j\omega C')} \quad (47)$$

$$Z_{ceq} = \sqrt{[R'+R_{pc}/a + j\omega(L'+L_{pc}/a)]/(G'+j\omega C')} \quad (48)$$

and $[T_{eq}]$ matrix is simply:

$$[T_{eq}] = \begin{bmatrix} e^{\gamma_{eq}(\omega)d} & 0 \\ 0 & e^{-\gamma_{eq}(\omega)d} \end{bmatrix} \quad (49)$$

VII. Effects of Pupin's Coils Inclusion

To illustrate an influence of inductance coils inclusion on line transmission and bandwidth we choose an example of cable used for early telephone transmission line. The primary parameters of that cable were: $R'=14.2$ Ω/mile, $L'=2$ mH/mile, $G'=24$ μS/mile, $C'=138$ nF/mile and the cable length was $d=42$ miles. It is obvious that the cable with such parameters is far away from satisfying Heaviside criterion. For this reason the cable was periodically loaded with discrete coils every one mile ($a=1$ mile, $N=42$). Inductance and losses of used coils were $L_{pc}=98$ mH and $R_{pc} = 6$ Ω, respectively. Per-unit length inductance required for fulfilment of Heaviside criterion for loaded line is $L_H'=116.15$ mH/mile.

Coefficient of transmission for original unloaded cable (solid line) and Pupinized cable (dashed line) is shown in Fig.16 at frequency range up to 4 kHz (voice band). At these relatively low frequencies, the variation of primary per-unit length parameters with frequency can be neglected. The wavelength corresponding to the maximum frequency of interest (in this case $f_{\max}=4$ kHz) can be found from (37) as $\lambda_{\min}=15$ miles which means that Pupin's rule of fifteen coils per minimum wavelength ($n=15$) is applied. In that case, total inductance per mile according to (36) is $L_t'=100$ mH/mile whereas the per-unit length resistance increases to 20.2 Ω/mile, so that Heaviside criterion fulfilment factor of Pupinized cable is $l_t \approx 0.86$. For comparison purpose, the transmission of equivalent uniform line of the same per unit length parameters as Pupinized cable is shown on the Fig.16 as well (dotted line). It is slightly below the transmission of uniform line that fully satisfies Heaviside criterion, i.e. $l_t=1$ (dash-dotted line).

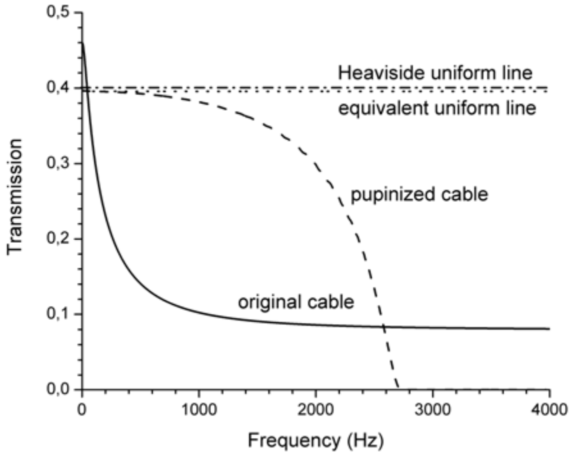


Fig.16. Transmission characteristics of original cable (solid line), Pupinized cable (dashed line), equivalent uniform line (dotted line) and Heaviside uniform line (dash-dotted line).

Fig.16 shows clearly why original cable was not suitable for baseband audio signal transmission. The same cable with discrete coils inserted in each mile section improves significantly the low frequency response. However, inclusion of coils changes transmission line structure and introduces periodic reflections resulting in a low-pass characteristics and limited bandwidth. From the filter theory, the cut-off frequency can be estimated as [8]:

$$f_{cut} \approx 1 / \left(a\pi \sqrt{(L'+L_{pc}/a)C'} \right) \quad (50)$$

and in this case is $f_{cut}=2.709$ kHz which can be also seen on Fig.16. It should be pointed out that we based our calculations on assumption that Pupinized line is terminated with impedance Z_p which is close to matched termination:

$$Z_p = \sqrt{(L'+L_{pc}/a)/C'} \quad (51)$$

It is worthwhile to investigate what would be the transmission of Pupinized cable if we keep the total per-unit length inductance constant but change the value of inductance of discrete coils and number of coils per minimum wavelength (i.e. coils distance). For simplicity reasons, coils losses are neglected ($R_{pc}=0$ Ω) and the Heaviside criterion fulfilment factor is chosen to be $l_t=1.0$ so that $L_t'=L_H'=81.65$ mH/mile. The coefficient of transmission of Pupinized cable for different number of coils per minimum wavelength, $\lambda_{\min}=15$ miles, is shown in Fig.17. Inductance of used coils, for different values of n, can be easily found from (38) ($L_{pc}=79.65$ mH, 39.825 mH and 19.9125 mH, for $n=15$, $n=30$ and $n=60$, respectively).

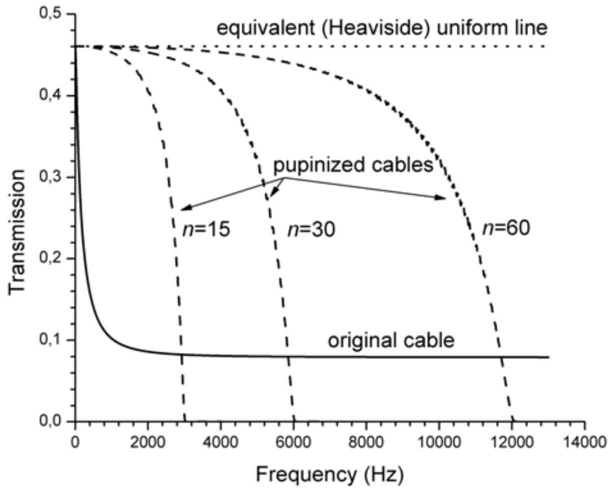


Fig.17. Transmission characteristics of Pupinized cable for different number of coils per wavelength and l_t=1.0.

Much wider bandwidth is achieved by increasing the number of coils per wavelength. Estimation of around twenty coils per minimum wavelength can be made in order to have a satisfying level of cable transmission within the voice band. As expected, Pupinized cable with n=60 coils per wavelength behaves, at the frequency range up to 4 kHz, almost as equivalent uniform line whose per-unit length inductance is equal to Heaviside's one. Similar conclusions can be provided for transmission coefficient of Pupinized cable with fulfilment factor l_t=0.5 (L_t'=0.5L_H'=40.825 mH) with one exception. Pupinized cable with any number of coils per minimum wavelength can not achieve, anywhere in considered frequency range, the level of transmission of equivalent uniform line that fully satisfies Heaviside criterion (Fig.18).

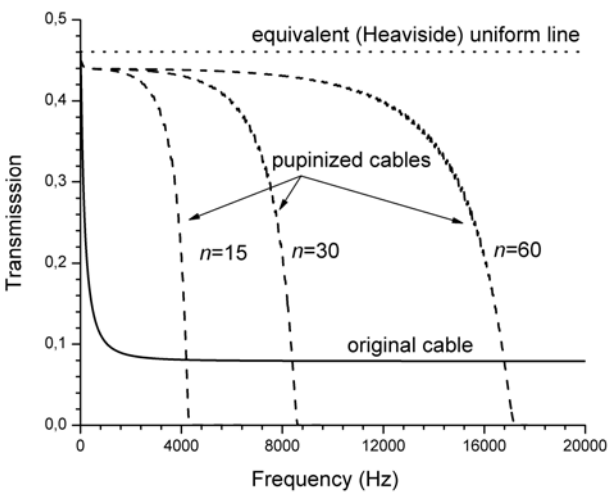


Fig.18. Transmission characteristics of Pupinized cable for different number of coils per wavelength and l_t=0.5.

VIII. Conclusion

In 1899 Pupin presented a new technique for construction of so-called Pupinized lines, based on theory he developed earlier and experimentally verified it. Idea to reduce attenuation and increase bandwidth of telephone cables was not new but Pupin was first to find, using complicated calculations, a solution for efficient increase of per-unit-length inductance and therefore efficient transmission of telephone signal through long cables and overhead lines. For his work, Pupin was awarded all over the world and he was named, among some other researchers, as a pioneer of long-distance telephone. Also, Pupin's theoretical and experimental work is an example of approach used today in modern research.

In this paper, after the presentation of Pupin's theory of propagation through unloaded and loaded telephone lines and its experimental verification, influence of periodically inserted inductance coils on cable transmission characteristics is analysed by the use of full wave matrix approach. Pupin's rule of equivalence represented in the form of appropriate number of coils per wavelength corresponding to maximum frequency of interest is challenged from the cable bandwidth and level of transmission point of view.

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