

# Formulation of Dynamics Equation for Electric Circuits with Computer Algebra Systems

Dejan V. Tošić

**Abstract**—A novel algorithm for automated computer-aided formulation of the state dynamics equation for lumped linear time-invariant electric circuits is presented. The algorithm is explained by an illustrative example. An original software package, written in *Mathematica*, is proposed for implementation of the algorithm. The new algorithm and the corresponding software implementation are intended for practitioners, researchers, educators and students, who design, explore, or evaluate electric circuits.

**Index Terms**—computer algebra system, electric circuit, state space analysis, symbolic analysis

## I. INTRODUCTION

In any electric circuit characterization one should seek the minimum mathematical description that leads to understanding and an intelligent use of the device. For general circuit studies the state variable qualifies admirably; the state-variable characterization provides the full complement of required information about the circuit in question in a most convenient form. The simple qualitative measures of circuit behavior, passivity, time-invariance, linearity, reciprocity, and stability are often easily discussed in terms of the state-variable characterization. [1]

Roughly speaking, the state of a system may be considered to be the minimal amount of information necessary at any time to characterize completely any possible future behavior of the system. For present purposes, the states become the set of independent initial conditions (or a nonsingular linear transformation of them) which the circuit can support. Thus, the states (initial conditions at time  $t_0$ ) plus the excitation (from time  $t_0$  onward) completely determine the response (from time  $t_0$  onward) for circuits that can be characterized by state variables. [1]

State-variable analysis or state-space analysis, as it is sometimes called, is a matrix-based approach that is used for analysis of circuits containing time-varying elements as well as nonlinear elements. The state of a circuit is defined as a set of a minimum number of variables

associated with the circuit; knowledge of these variables along with the knowledge of the input will enable the prediction of the currents and voltages in all circuit elements at any future time. Only capacitors and inductors are capable of storing energy in a circuit, and so only the variables associated with them are able to influence the future condition of the circuit. The voltages across the capacitors and the currents through the inductors may serve as state variables. [2]

There are many advantages in using the state equations [3]: (a) there is an enormous amount of mathematical knowledge for solving such equations while the equations by themselves can be derived from formal topological properties of the circuit, using the matrix approach, (b) it can be easily and naturally extended to nonlinear and time-varying or switched circuits and is, in fact, the approach most often used in characterizing such circuits, (c) a simple systematic method for writing such equations can be formulated by using the graph theory, and (d) it may be easily programmed for a numerical and symbolic solution with appropriate computer software.

The concept of state variables, or just state, satisfies two basic conditions of circuit analysis [3]:

1) If at any time, say  $t_0$ , the state is known (which is the initial condition or initial state), then the state equations uniquely determine the state at any time  $t > t_0$  for any given input. In other words, given the state of the circuit at time  $t_0$  and all the inputs, the behavior of the circuit is completely determined for all  $t > t_0$ .

2) The state and the input uniquely determine the value of the remaining circuit variables.

The initial conditions, or initial state of the circuit, are actually the initial values of the capacitor voltages and inductor currents, which usually can be independently specified in the circuit, i.e. their values just after  $t_0$  are determined by their values just before  $t_0$ .

All circuits to be considered here are to be lumped, linear, time-invariant, and finite so that only finite dimensional state spaces need be discussed. It is without loss of generality that the general circuit in question is assumed to be connected.

This paper presents a new algorithm, and the corresponding software implementation in *Mathematica* [4,5], for automated computer-aided formulation of the first-order differential state-space equations referred to as the circuit dynamics equation.

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## II. USE OF COMPUTER ALGEBRA SYSTEMS FOR FORMULATION OF THE STATE-SPACE EQUATIONS

Computers were initially developed to expedite numerical calculations. A newer, and in the long run, very fruitful field is the manipulation of symbolic expressions. When these symbolic expressions represent mathematical entities, this field is generally called computer algebra. [6]

The current symbolic computation environments are extremely powerful in doing symbolic and mixed symbolic-numeric mathematics for technical computing. They can be used for the efficient and effective search for optimal solutions. [7,8]

Computer algebra systems can be used to formulate general circuit equations and to rearrange them to the state-space equations.

The essence of the proposed symbolic algorithm for formulation the state-space equations is based on the modified nodal analysis (MNA) [9,10]. The MNA equations are extended by introducing one variable and one equation for each dynamic element.

Consider dynamic elements with standard reference directions as shown in Fig. 1. For each capacitor of a capacitance  $C$ , the element equation is

$$i_C = C \frac{dv_C}{dt} \quad (1)$$

and can be written in terms of new variables as

$$C D_{vC} = i_C, \quad D_{vC} = \frac{dv_C}{dt}, \quad v_k - v_m = v_C, \quad (2)$$

where  $D_{vC}$  is the new capacitor variable,  $v_k$  is the voltage between node  $k$  and the reference node, and  $v_m$  is the voltage between node  $m$  and the reference node.

Similarly, for each inductor of an inductance  $L$ , the element equation is

$$v_L = L \frac{di_L}{dt} \quad (3)$$

and can be written in terms of new variables as

$$L D_{iL} = v_L, \quad D_{iL} = \frac{di_L}{dt}, \quad v_k - v_m = v_L, \quad (4)$$

where  $D_{iL}$  is the new inductor variable.

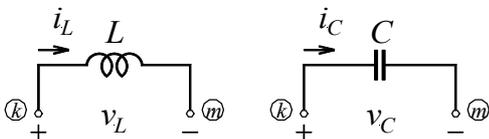


Fig1 1. Dynamic elements with standard reference directions.

In nodal analysis, Kirchhoff's current law (KCL) is used to write equations at each node in terms of the nodal voltages and element values. The equations are written for one node at a time, and the node voltages are unknown variables. In computer aided design (CAD), it is useful to consider one element at a time and to develop the matrix equations on this basis. [11]

Assume that the nodes in a circuit are labeled by consecutive integers from 1 to  $n$  such that there are the  $n$  nodal voltages in the circuit. The reference node is usually labeled zero.

In the proposed algorithm the capacitor will affect the KCL equations at nodes  $k$  and  $m$  in the following way:

$$\text{KCL at node } k : \dots + C D_{vC} + \dots = 0 \quad (5)$$

$$\text{KCL at node } m : \dots - C D_{vC} + \dots = 0 \quad (6)$$

and the ellipses (...) denote contributions from other elements connected to that node. This increases the number of unknown variables by one (by  $D_{vC}$ ). Hence, another equation is needed so that the number of unknowns and the number of equations are the same and can be solved. The additional equation is the following:

$$v_k - v_m = v_C \quad (7)$$

and the capacitor voltage  $v_C$  is treated as a known quantity.

Similarly, the inductor will affect the KCL equations as follows:

$$\text{KCL at node } k : \dots + i_L + \dots = 0 \quad (8)$$

$$\text{KCL at node } m : \dots - i_L + \dots = 0 \quad (9)$$

and the inductor current  $i_L$  is treated as a known quantity.

The additional equation is the following:

$$v_k - v_m - L D_{iL} = 0 \quad (10)$$

and it contains the new variable ( $D_{iL}$ ). If coupled inductors exist, equation (10) should be updated accordingly.

Consider a general circuit of  $n+1$  nodes,  $n_C$  capacitors and  $n_L$  inductors, that is,  $n_D = n_C + n_L$  dynamic elements. The MNA equations will have additional  $n_D$  variables and additional  $n_D$  equations. Treating the capacitor voltages and inductor currents as known quantities, the MNA equations can be solved and the  $D_{vC}$ ,  $D_{iL}$  variables can be obtained as linear combinations of the  $v_C$ ,  $i_L$  quantities and excitations (inputs).

The above algorithm is implemented in *Mathematica* as a SSALEC program (State-space Symbolic Analysis of Electric Circuits), which complements the SALECAS software reported in [12].

SSALEC receives a netlist based description of a circuit and formulates the state-space differential equations. In addition, it returns the solution for MNA variables (node voltages and some currents).

## III. EXAMPLE OF SSALEC OPERATION

Consider an RLC circuit shown in Fig. 2 [13, p347].

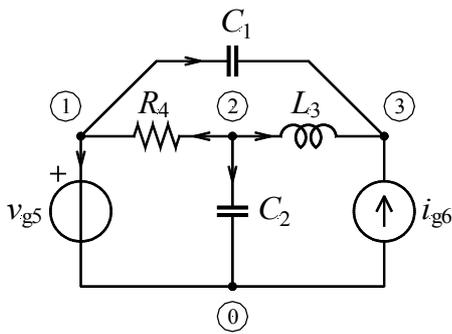


Fig. 2. Example RLC circuit.

Netlist for the example RLC circuit follows:

```
(* RLC.cir Example RLC Circuit *)
NumberOfNodes = 3
component[1] = {"C", "C1", 1, 3, C1}
component[2] = {"C", "C2", 2, 0, C2}
component[3] = {"L", "L3", 2, 3, L3}
component[4] = {"R", "R4", 2, 1, R4}
component[5] = {"V", "vg5", 1, 0, vg5}
component[6] = {"I", "ig6", 0, 3, ig6}
NumberOfComponents = 6
```

The corresponding SSALEC operation and the symbolic analysis results are shown in Fig. 3. Note that there are three differential state-space equations.

```
SSALEC["n2008_RLC.cir"];

(* RLC.cir Example RLC Circuit *)
NumberOfNodes = 3
component[1] = {"C", "C1", 1, 3, C1}
component[2] = {"C", "C2", 2, 0, C2}
component[3] = {"L", "L3", 2, 3, L3}
component[4] = {"R", "R4", 2, 1, R4}
component[5] = {"V", "vg5", 1, 0, vg5}
component[6] = {"I", "ig6", 0, 3, ig6}
NumberOfComponents = 6

V1 = vg5
V2 = vC2
V3 = -vC1 + vg5

J4vg5 = ig6 + iL3 +  $\frac{vC2}{R4} - \frac{vg5}{R4}$ 

D1vC1 =  $-\frac{ig6}{C1} - \frac{iL3}{C1}$ 
D2vC2 =  $-\frac{iL3}{C2} - \frac{vC2}{C2 R4} + \frac{vg5}{C2 R4}$ 
D3iL3 =  $\frac{vC1}{L3} + \frac{vC2}{L3} - \frac{vg5}{L3}$ 
```

Fig. 3. SSALEC symbolic formulation of the state-space equations for the example RLC circuit.

The results of symbolic analysis performed by SSALEC can be written in the conventional matrix form as follows:

$$\mathbf{x} = \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L3} \end{bmatrix} \quad (11)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \frac{-1}{C_1} \\ 0 & \frac{-1}{C_2 R_4} & \frac{-1}{C_2} \\ \frac{1}{L_3} & \frac{1}{L_3} & 0 \end{bmatrix} \quad (12)$$

$$\mathbf{B} = \begin{bmatrix} 0 & \frac{-1}{C_1} \\ \frac{1}{C_2 R_4} & 0 \\ \frac{-1}{L_3} & 0 \end{bmatrix} \quad (13)$$

$$\mathbf{u} = \begin{bmatrix} v_{g5} \\ i_{g6} \end{bmatrix} \quad (14)$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (15)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \quad (16)$$

The  $\frac{d\mathbf{x}}{dt}$  equation (15) is called the state dynamics equation, and the  $\mathbf{y}$  equation (16) is called the output equation.  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ , and  $\mathbf{D}$  are appropriately dimensioned coefficient matrices.

The state variables are commonly expressed as a vector  $\mathbf{x}$ . SSALEC arranges state variables according to the ordering of dynamic elements (capacitors and inductors) in the netlist.

The input source variables are expressed as a vector  $\mathbf{u}$ .

The non-state-variables are expressed as a vector  $\mathbf{y}$ .

This set of equations, where the derivative of state variables is expressed as a linear combination of state variables and forcing functions, is said to be in normal form:

$$\begin{aligned} \frac{dv_{C1}}{dt} &= \frac{-1}{C_1} i_{L3} + \frac{-1}{C_1} i_{g6} \\ \frac{dv_{C2}}{dt} &= \frac{-1}{C_2 R_4} v_{C2} + \frac{-1}{C_2} i_{L3} + \frac{1}{C_2 R_4} v_{g5} \\ \frac{di_{L3}}{dt} &= \frac{1}{L_3} v_{C1} + \frac{1}{L_3} v_{C2} + \frac{-1}{L_3} v_{g5} \end{aligned} \quad (17)$$

#### IV. CONCLUSION

This paper has reviewed the very basic concepts of the state-variable characterization of lumped linear electric circuits. The state-space equations have been formulated by use of a computer algebra system. A new algorithm for automated computer-aided symbolic computation of the state dynamics equation has been proposed. A *Mathematica* program SSALEC has been presented as a software implementation of the proposed algorithm. The SSALEC operation has been demonstrated step-by-step by an illustrative example.

The value of symbolic state-space analysis could be beneficial in both industry and academia. In industry it could be used as an aid in the design of systems and circuits. In academic institutions it could be found useful as an instructional aid.

The future research efforts are directed towards a more general algorithm for the formulation of state-space equations with the algebraic degeneration – taking into account capacitor loops and inductor cut-sets.

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