Measurement error estimation of light spot position on a quadrant photodiode

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Abstract: **New closed form expressions for estimating the errors of displacement signal and light spot position, for the position-sensitive detector (PSD) with quadrant photodiode (QPD), is presented. Both errors increase towards the limits of the measurement span, but the error of spot position is changing faster.**

Keywords: **Position-sensitive detector, Quadrant photodiode.**

I. Introduction

Theory of tracking accuracy of laser systems, including analyses of angular measurement span, angle estimation bias, and angle estimation variance, is given in [1]. Positioning resolution of PSD in high background illumination is presented in [2]. In both papers it is proved that signal-to-noise ratio (SNR) limits accuracy of positioning in PSD with QPD. The probability density function of the displacement signal in PSD with QPD, for uncorrelated Gaussian noise between quadrant's currents, is given in [3]. In this paper displacement signal, variance of displacement signal, error of displacement signal and position error for PSD with QPD are investigated. The error of displacement signal using both displacement signal and probably density function of displacement signal is derived. The position error by the error of displacement signal is analyzed.

II. Displacement signal

Consider a light spot of radius r falling on a QPD with radius a as shown in Fig. 1. The position of the photodiode center is at (0,0) and the center of the spot at (x, y), where x, and y are displacement positions of the spot center along the X- and Y- axes, respectively. Displacement signals depend on the spot positions in both directions. These signals can be obtained by processing the output currents of four photodiode quadrants. The normalized displacement signal ε_x in the X direction is:

$$\varepsilon_x = \frac{(i_I + i_{II}) - (i_{III} + i_{IV})}{(i_I + i_{II}) + (i_{III} + i_{VI})} = \frac{u-v}{u+v}, \qquad (1)$$

where: i_i is the current of i-th quadrant (i=I, II, III and IV), $i_i = \Re p_i P_0$, where P_0 is the total received flux, p_i is proportional to the area of the spot falling on i-th quadrant ($0 \le p_i \le 1$), $\Re$ is the responsivity of photodiode, u and v are the variables which represent signals taken pairwise. Similar formula can be obtained for the Y direction.

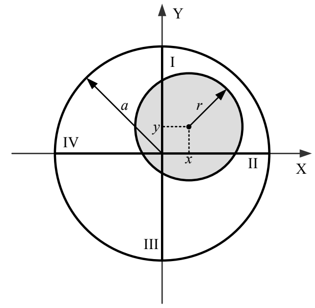

Fig. 1. Geometry of the light spot position on the QPD surface.

Displacement signal depends on the shape of spot and irradiance distribution on the photodiode surface. By simple geometrical calculations, displacement signal in the X direction, for uniform irradiance distribution and circular shape of the spot, is obtained in the form:

$$\varepsilon_x = \frac{2}{\pi}\left[\frac{x}{r}\sqrt{1-\frac{x^2}{r^2}} + \arcsin\left(\frac{x}{r}\right)\right], \quad |x| \le r. \qquad (2)$$

The displacement signal given in Eq. (2) is a non-linear function of x/r and requires that $|x| \le r$, and $a \ge 2r$.

III. Variance of displacement signal

Noise, which is inherent to any electronic device, is one of the most significant causes of the angle measurement error in pulsed laser tracking systems. According to Eq. (1) and taking into consideration the equivalent noise sources, we can write:

$$\varepsilon_x = \frac{(\bar{u} + U_n) - (\bar{v} + V_n)}{(\bar{u} + U_n) + (\bar{v} + V_n)}, \qquad (3)$$

where $u = \bar{u} + U_n, v = \bar{v} + V_n$; $\bar{u}$ and $\bar{v}$ are mean values; U_n and V_n are the fluctuations of u and v, respectively.
If we rearrange the Eq. (3) and assuming that $U_n + V_n << \bar{u} + \bar{v}$ we will get:

$$\varepsilon_x \approx \frac{\bar{u} - \bar{v}}{\bar{u} + \bar{v}} + 2\frac{\bar{v}U_n - \bar{u}V_n}{(\bar{u} + \bar{v})^2} = \bar{\varepsilon}_x + \varepsilon_{xn}. \qquad (4)$$

The first part of Eq. (4) represents the mean value of the relative light spot displacement and the second part represents the random in displacement measuring caused by noise. For this measurement error ε_{xn}, due to the noise of displacement signal, we can write:

$$\varepsilon_{xn} \approx \frac{2\bar{v}}{(\bar{u} + \bar{v})^2}U_n - \frac{2\bar{u}}{(\bar{u} + \bar{v})^2}V_n. \qquad (5)$$

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As the noise sources in each receiving channel are independent, for the variance σ_ε^2 of the relative displacement measurement the following is valid:

$$\sigma_\varepsilon^2 \approx \frac{4\bar{v}^2}{(\bar{u}+\bar{v})^4}\sigma_u^2 + \frac{4\bar{u}^2}{(\bar{u}+\bar{v})^4}\sigma_v^2, \qquad (6)$$

where σ_u^2 and σ_v^2 are the variances of the U_n and V_n, respectively and $\sigma_u^2 = \sigma_v^2 = 2\overline{i_n^2}$, where $\overline{i_n^2}$ is the variance of the noise current in each receiving channel. Now, Eq. (6) can be rearranged in:

$$\sigma_\varepsilon^2 \approx 8\frac{\bar{u}^2+\bar{v}^2}{(\bar{u}+\bar{v})^4}\overline{i_n^2}. \qquad (7)$$

As for $\bar{u}$ and $\bar{v}$, $S_S = \bar{u}+\bar{v}$ is valid, where S_S is the signal in the summing channel, the variance σ_ε^2 of the relative displacement measurement becomes:

$$\sigma_\varepsilon^2 \approx 8\frac{\bar{u}^2+(S_S-\bar{u})^2}{S_S^4}\overline{i_n^2}. \qquad (8)$$

The variance σ_ε^2, from Eq. (8), has the minimum value for $\bar{u}=\bar{v}=S_S/2$, so we can write:

$$\left(\sigma_\varepsilon^2\right)_{\min} \approx \frac{4\overline{i_n^2}}{S_S^2} = \frac{1}{SNR_\Sigma} \qquad (9)$$

where SNR_Σ is signal-to-noise ratio in the summing channel.

As it can be seen, the minimal variance of the relative displacement measurement is achieved when $\bar{u}=\bar{v}=S_S/2$ is valid, i.e. when the light spot center is at the QPD center. The parameters $\bar{u}$ and $\bar{v}$ satisfies the following inequalities: $\bar{u} \le S_S$ and $\bar{v} \le S_S$. Because of that and according to Eq. (8), the maximal value of the σ_ε^2 is achieved when $\bar{u} = S_S$ and $\bar{v} = 0$ (or $\bar{v} = S_S$ and $\bar{u} = 0$) is fulfilled, i.e. when the light spot center is at the measurement span margin ($x=\pm r$ and/or $y=\pm r$), so we have:

$$\left(\sigma_\varepsilon^2\right)_{\max} \approx \frac{2}{SNR_\Sigma} \qquad (10)$$

The identical expressions are valid for the maximal standard deviation of estimated displacement along the Y-axis.

As it can be seen, the maximal standard deviation of measured displacement signal is $\sqrt{2}$ times bigger than its minimal value. For us of interest is the minimal error in positioning and tracking, because in the case when the centers of the light spot and the QPD are matched, we have a good positioning and tracking.

III. Error of displacement signal

Derived variance, in Eq. (8), of displacement signal is not useful for analyzing error of displacement signal in all its values. The error of displacement signals used probably density function of signal positioning is derived.

Displacement signal uncertainty of a PSD with QPD can be estimated from the probability density function of the displacement signal in the presence of noise. It has been assumed that each quadrant of the photodiode generates Gaussian distribution noise with zero mean value and variance σ_n^2 [3]. Then *u* and *v* in Eq. (1) represent total signals from pair quadrants of the photodiode, where $u = \bar{u} + U_n, v = \bar{v} + V_n$; $\bar{u}$ and $\bar{v}$ are mean values; U_n and V_n are the fluctuations of *u* and *v*, respectively. The probability density function of the displacement signal $f(\varepsilon)$, is given in [3].

On the base of known theory, in [4] is derived probability density function for the error signal in a form

$$f(\varepsilon) = \frac{\sqrt{1-\rho^2}}{\pi(1+\varepsilon^2-2\rho\varepsilon)}\cdot\exp\left(-\frac{\bar{u}^2+\bar{v}^2-\rho(\bar{u}^2-\bar{v}^2)}{2\sigma_p^2(1-\rho^2)}\right)\cdot \left[1+\sqrt{\frac{\pi}{2}}\cdot B\left(erf(\frac{B}{\sqrt{2}})\right)\exp\frac{B^2}{2}\right] \qquad (11)$$

where: ρ is the correlation between U_n end V_n, $\sigma_p^2=2\sigma_n^2$, and B is defined as

$$B = \frac{\bar{u}(1+\varepsilon)+\bar{v}(1-\varepsilon)-\rho\left(\bar{u}(1+\varepsilon)+\bar{v}(\varepsilon-1)\right)}{\sqrt{2}\sigma_p\sqrt{(1-\rho^2)(1+\varepsilon^2-2\rho\varepsilon)}}$$

Equation (11) shows that probability density function *(pdf)* of error signal is approximately Gaussian. Analysis of *pdf* is given in [3] and [4]. In [4] was shown that the value of ρ changes both the maximum and width of *pdf*. Maximum width and minimum amplitude of *pdf* was obtained for correlation coefficient equal to zero. That means, the worst case for the position error is when noise between pair quadrants is uncorrelated (ρ=0) .

An useful way to find the position error that the mean value pair (*u*,*v*) substitute with mean value of the error signal ($\bar{\varepsilon}$). The mean value of the error signal , $\bar{\varepsilon}$ is:

$$\bar{\varepsilon} = \frac{\bar{u}-\bar{v}}{\bar{u}+\bar{v}} \qquad (12)$$

Signal-to-noise ratio in the channel sum is

$$SNR = \frac{(\bar{u}+\bar{v})^2}{4\sigma_n^2} \qquad (13)$$

After substituting (12) and (13) in (11) and rearranging equation for *B*, for ρ=0, is obtained following expression

$$B = \frac{\bar{u}(1+\varepsilon)+\bar{v}(1-\varepsilon)}{\sqrt{2}\sigma_p\sqrt{1+\varepsilon^2}} = \frac{1+\varepsilon\bar{\varepsilon}}{\sqrt{1+\varepsilon^2}}\sqrt{SNR} \qquad (14)$$

Combining (12) and (13) we can obtain

$$\frac{\bar{u}^2+\bar{v}^2}{4\sigma_n^2} = \frac{1}{2}SNR(1+\bar{\varepsilon}^2) \qquad (15)$$

We have simplified this probability density function (11), for ρ=0, and using an approximation for the complementary error function $erfc(x) \approx \exp(-x^2)/(\sqrt{\pi}x)$. The new approximate probability density function is thus obtained in the form:

$$f(\varepsilon) \approx \sqrt{\frac{SNR}{2\pi}} \frac{1+\varepsilon\bar{\varepsilon}}{\left(1+\varepsilon^2\right)^{3/2}} \exp\left[-\frac{SNR(\varepsilon-\bar{\varepsilon})^2}{2\left(1+\varepsilon^2\right)}\right] \quad (16)$$

The relative error of approximation given in Eq. (16) is less than 0.8% for the worst case $\bar{\varepsilon}=0$, and small *SNR*=5, and decreases rapidly with increasing *SNR*. From $f(\varepsilon)$ one can calculate the probability that the value of ε falls in the range $\bar{\varepsilon}-\Delta\varepsilon \le \varepsilon \le \bar{\varepsilon}+\Delta\varepsilon$. The value of $\Delta\varepsilon$ is usually calculated for probability that the displacement signal is in the expected range for 50% (CEP - the Circular Error Probability) of all possible signals around the mean displacement signal. The probability is calculated from integral:

$$P_p = \int_{\bar{\varepsilon}-\Delta\varepsilon}^{\bar{\varepsilon}+\Delta\varepsilon} f(\varepsilon)d\varepsilon = \frac{1}{\sqrt{\pi}} \int_{w_1}^{w_2} \exp\left(-w^2\right)dw, \quad (17)$$

where $\Delta\varepsilon$ is defined as the error of the displacement signal, and new variable $w=\left[(\varepsilon-\bar{\varepsilon})/\sqrt{1+\varepsilon^2}\right]\sqrt{SNR/2}$.

Substituting Eq. (16) into Eq. (17), after integration, we obtain in closed form the probability of all possible values ε as the function of $\Delta\varepsilon$, $\bar{\varepsilon}$ and *SNR*:

$$P_p = \frac{1}{2}\left\{ \begin{aligned} & erf\left[\sqrt{\frac{SNR}{2}} \frac{\Delta\varepsilon}{\sqrt{1+(\bar{\varepsilon}+\Delta\varepsilon)^2}}\right] + \\ & + erf\left[\sqrt{\frac{SNR}{2}} \frac{\Delta\varepsilon}{\sqrt{1+(\bar{\varepsilon}-\Delta\varepsilon)^2}}\right] \end{aligned} \right\}. \quad (18)$$

It can be shown, that for *SNR*>10, the arguments of the error functions allow further approximation of P_p from Eq. (18) leading to:

$$P_p \approx erf\left(\sqrt{\frac{SNR}{2}} \frac{\Delta\varepsilon}{\sqrt{1+\bar{\varepsilon}^2+\Delta\varepsilon^2}}\right). \quad (19)$$

The relative error of this approximation, in the worst case $\bar{\varepsilon}=1$, is less than 2.5%. By solving Eq. (19) for $\Delta\varepsilon$ we obtain the error of displacement signal in the form:

$$\Delta\varepsilon = \frac{\sqrt{1+\bar{\varepsilon}^2}}{\sqrt{\left(SNR/2C^2\right)-1}}, \quad (20)$$

where $C = erfinv(P_P)$.

As can be seen from Eq. (20), the maximum value of $\Delta\varepsilon$, for any given signal-to-noise ratio is $\sqrt{2}$ times larger for $\bar{\varepsilon}=1$ in comparison to the minimum value for $\bar{\varepsilon}=0$. The minimum value of $\Delta\varepsilon$ is inversely proportional to $\sqrt{SNR}$, for $SNR >> 2C^2$. The error of displacement signal given in Eq. (20), for constant probability P_p=0.5, for two values of SNR, as the function of the mean value of displacement signal is shown in Fig. 2.

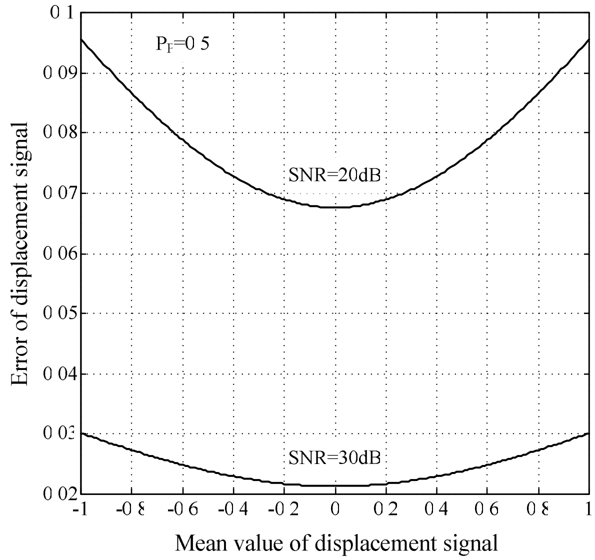

Fig. 2. Error of displacement signal as the function of its mean value, for P_p=0.5 and SNR=20dB and 30dB.

V. Position error

The position error is obtained from the derivative of the displacement signal with respect to *x*. The normalized position error in the *x* direction, for displacement signal given by Eq. (2) is obtained as:

$$\frac{\Delta x}{r} = \frac{\pi}{4} \frac{\Delta\varepsilon}{\sqrt{1-(x/r)^2}} \quad (21)$$

where $\Delta\varepsilon$ is the error of the displacement signal given by Eq. (20).

The normalized position error calculated from Eq. (21) and Eq. (20), as a function of normalized position *x/r*, is shown in Fig. 3.

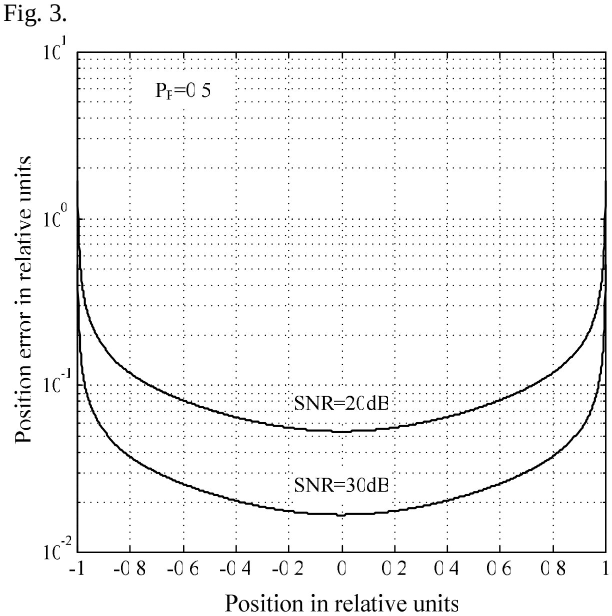

Fig. 3. Normalized position error as the function of normalized displacement, for P_p=0.5 and SNR=20dB and 30dB.

The normalized position error depends on *SNR* and *x/r*. The minimum position error is obtained when the center of the spot is in the center of photodiode, and increases with $|x|/r$, for constant *SNR*. From Fig. 3. it is seen that the position error rapidly increases at the limits of the measurement range $x/r=\pm 1$.

VI. SUMMARY

In this paper the error estimation in measurement of light spot position on a quadrant photodiode is presented. Influence of electronic noise, as one of the most significant cause of measurement error, was investigated. It was shown that measurement error is only slightly changed when the light spot center is approaching the measurement span limits. The error on the measurement span limits is only $\sqrt{2}$ times greater then when the light spot is at the QPD centre. The same result was obtained by both methods – analyzing the displacement signal by finding its variance and analyzing the probability density function of the displacement signal in the presence of noise.

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SADRŽAJ

U radu su dati novi zatvoreni oblici relacija za procenu greške pozicioniranja spota i greške signala pozicioniranja za pozicionim sensitivnim senzorom, sa kvadrantnom fotodiodom. Obe greške rastu sa pomeranjem spota od centra fotodiode, ali greška pozicioniranja brže raste naročito u okolini maksimalnog odstupanja.

GREŠKA POZICIONIRANJA SPOTA NA KVADRANTNOJ FOTODIODI

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